Network Medicine

Petri Nets: Properties

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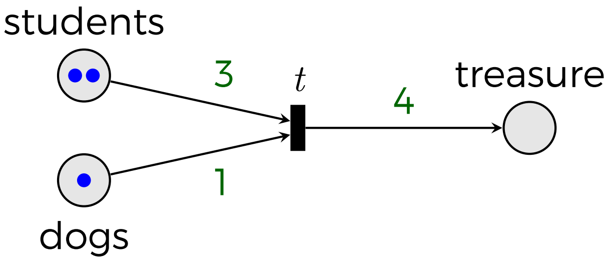
http://lacl.fr/~sivanov/doku.php?id=en:pn-biomodelling

Petri Nets: Reminder



$P = \{\text{students}, \text{dogs}, \text{treasure}\}$ $\quad T = \{t\}$

	(students, t)	(dogs, t)	(t, treasure)
W	3	1	4

	students	dogs	treasure
M_0	2	1	0

The Spirit of the Lecture

What? Formal descriptions of what we (would like to) see.

Why? To properly formulate queries to computers.

Feels like? A kind of mind games.

Outline

1. Behavioural Properties

2. Structural Properties

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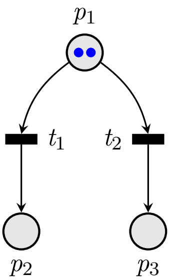
2. Structural Properties

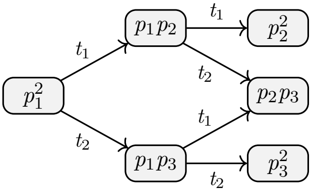
Behavioural Properties

Properties of the state graph.

- how does the net evolve?
- which markings may it attain?

May depend on the evolution mode.



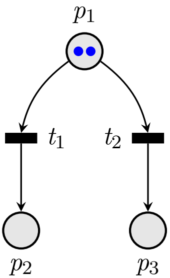


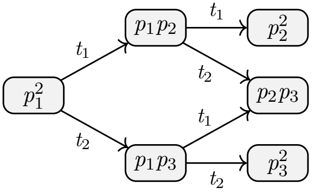
the state graph under asynchronous mode

Reachability

Given a marking (state) M, can the net reach it from its initial marking M_0?

Depends on the mode.





asynchronous mode

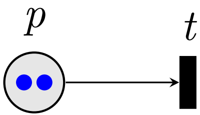
Markings p_1^2, $p_1 p_2$, $p_1 p_3$, p_2^2, $p_2 p_3$, p_3^2 are reachable.

Are there any markings this net cannot reach under asyn?

p_1^3, $p_1^2 p_2$, $p_2^2 p_3$, etc.

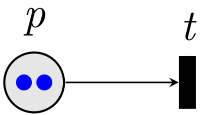
What is the reachability set? 1/3

Reachability set = all states listed in the state graph



What is the reachability set? $^1/_3$

Reachability set $=$ all states listed in the state graph



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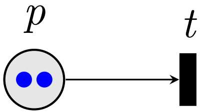

Answer: $\{p^2, p^1, p, \lambda\}$ $\leftarrow$ λ is the empty marking

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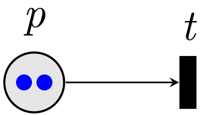
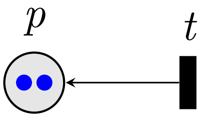


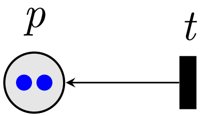
Answer: $\{p^2, p^1, p, \lambda\}$ $\leftarrow$ λ is the empty marking

Does the choice of the evolution mode matter?

What is the reachability set? 2/3



What is the reachability set? $^2/_3$



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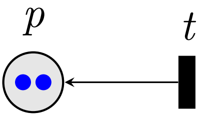
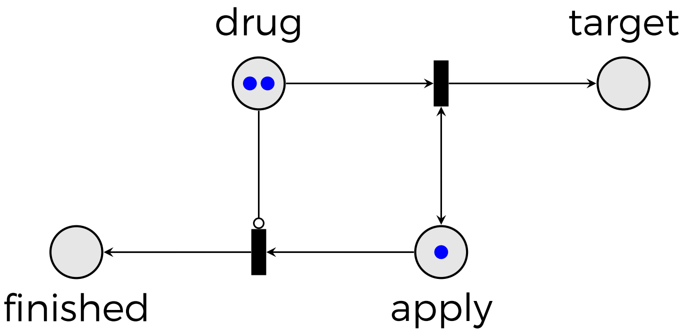


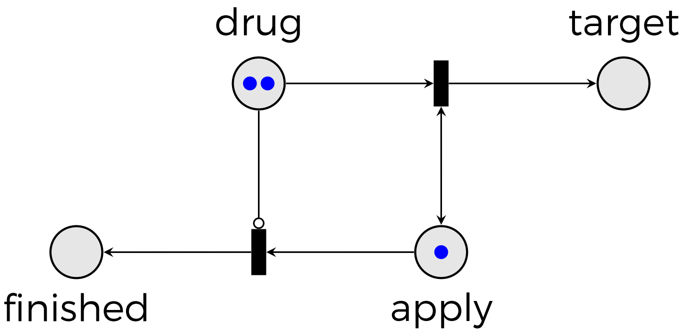
Answer: $\{p^k \mid k \in \mathbb{N}, k \geq 2\}$

Does the choice of the evolution mode matter?

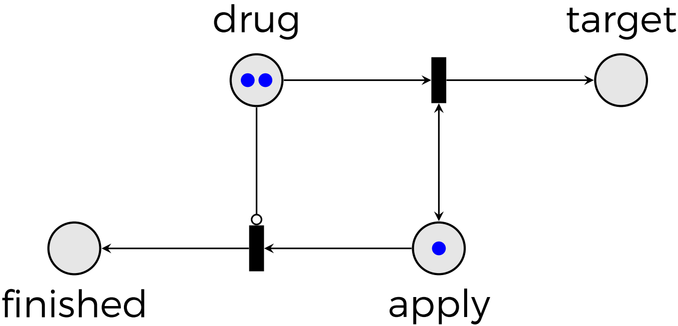
What is the reachability set? $^3/_3$



What is the reachability set? 3/3

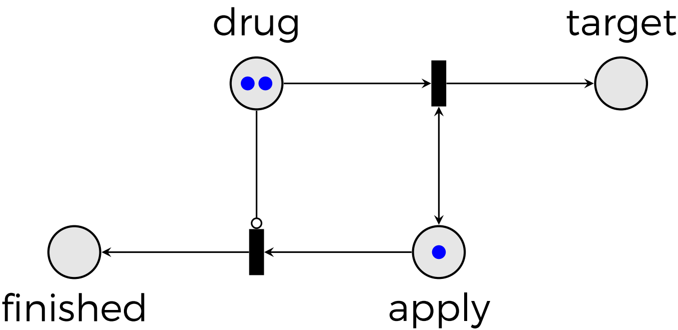


What is the reachability set? $^3/_3$



Answer: $\{d^2a, dat, at^2, ft^2\}$

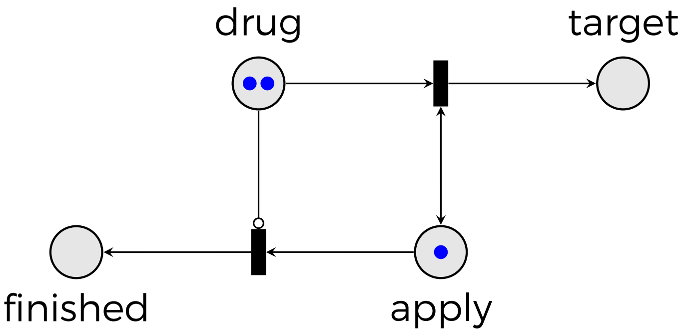
What is the reachability set? $^3/_3$



Answer: $\{d^2a, dat, at^2, ft^2\}$

What is the activity/phenomenon this net models?

What is the reachability set? $^3/_3$



Answer: $\{d^2a, dat, at^2, ft^2\}$

What is the activity/phenomenon this net models?

Does the choice of the evolution mode matter?

Sidenote: Asyn is Often Used in Modelling

The asynchronous mode well represents arbitrary interleaving of process interactions.

Ensuring a certain behaviour under the asynchronous mode means proper synchronisation.

Reachability is Hard

Reachability is decidable.

- $\exists$ a Turing machine deciding whether a marking is reachable or not.

Reachability is EXPSPACE-hard.

- A Turing machine needs at least exponential space on the band in order to decide whether a marking is reachable or not.
- Essentially, one needs to look over almost all of the reachability graph.

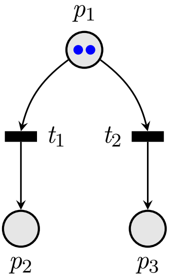
Coverability: "Lighter" Reachability

Given a marking M, can the net reach a marking M' such that M' covers M?

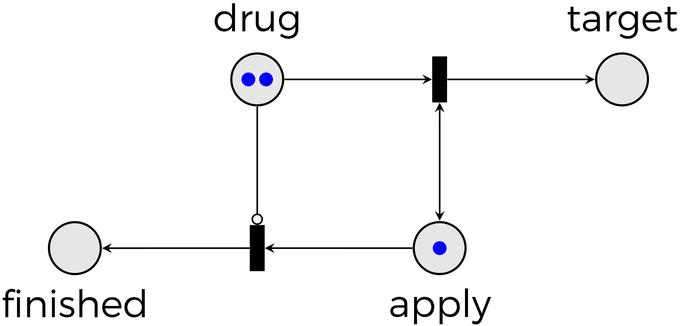
- M' covers M if all places in M' contain at least as many tokens as in M ($M' \geq M$).

Markings p_2 and p_3

- are coverable under both syn and asyn
- are not reachable



Is a Marking Coverable?

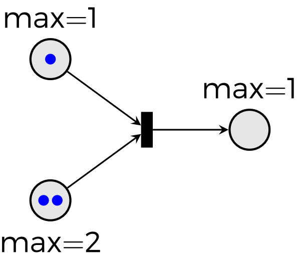


Which of these markings are coverable: dt, af, df?

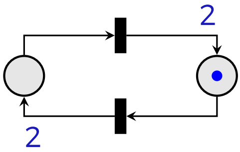
Boundedness

A Petri net is bounded if the number of tokens in every place never exceeds a fixed constant.

Bounded



Unbounded



The number of tokens in the net increases at every step.

Unboundedness $\implies$ Cycles?

Do all unbounded Petri nets have cycles?

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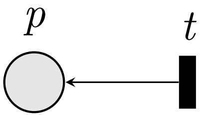
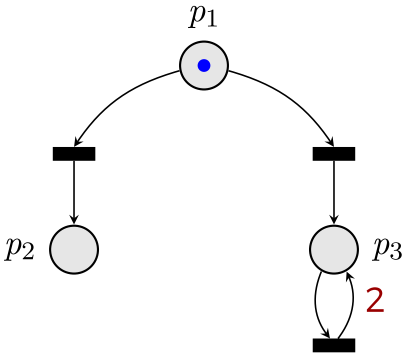
Do all unbounded Petri nets have cycles?



t produces new tokens all the time.

Is This Net Unbounded? $^1/_2$



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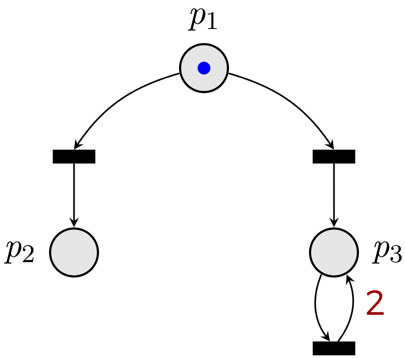

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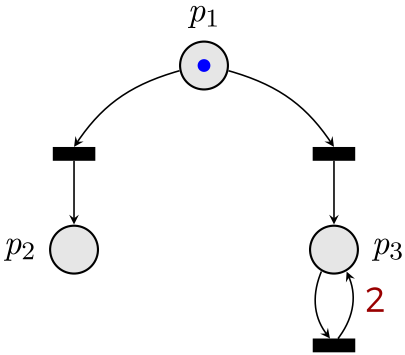
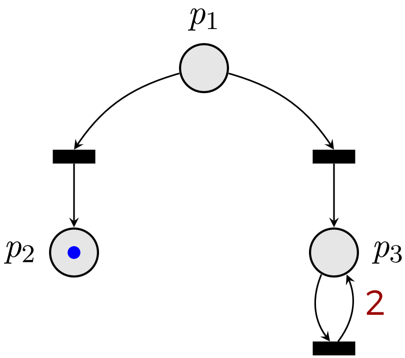


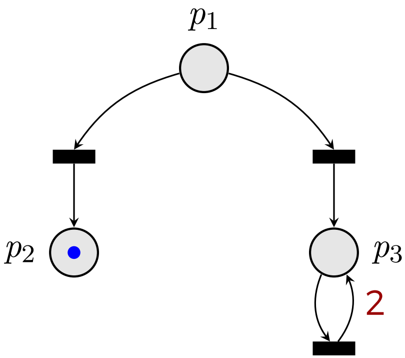
Answer: Yes

The token may get into p_3.

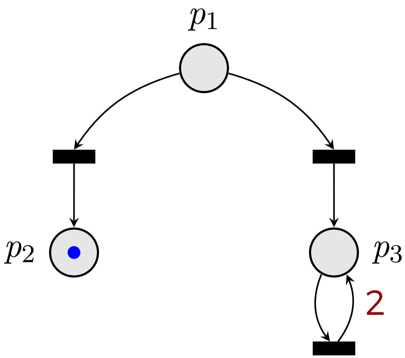
Is This Net Unbounded? $^2/_2$



Is This Net Unbounded? $^2/_2$



Is This Net Unbounded? $^2/_2$



Answer: No

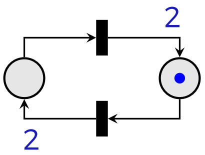
No tokens ever get into p_3.

Liveness

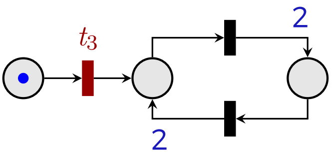
A Petri net is live if, starting from any reachable marking, any transition in the net can be eventually fired.

$\forall M \in$ Reachable(M_0), $\forall t \in$ Transitions,
$\exists M' \in$ Reachable(M) such that t is enabled at M'.

Live



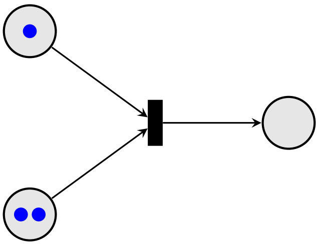
Not live



t_3 may only fire once.

Deadlocks

Is this net live?



Deadlocks

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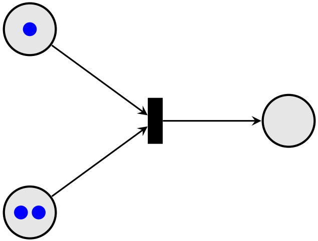
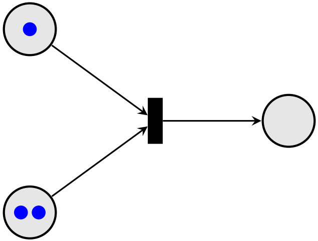


Deadlocks

Is this net live?



Answer: No

Deadlock = a state in which no transitions are enabled.

What are the deadlocks of this net?

Summary of Behavioural Properties

- Reachability and coverability
  - Can a given marking be reached/covered?
- Boundedness
  - Is there a fixed upper bound on the number of tokens in all places?
- Liveness and deadlocks
  - Can any transition fire arbitrarily often?

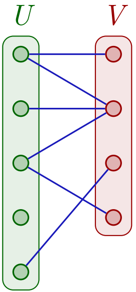
Outline

1. Behavioural Properties

2. Structural Properties

Bipartite Graphs

A graph is bipartite if its vertices can partitioned into two sets U and V such that every edge connects a vertex from U to a vertex from V (or a vertex from V to a vertex from U).



no edges within U or V

https://en.wikipedia.org/wiki/Bipartite_graph

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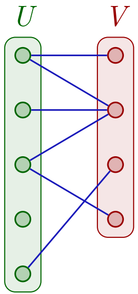

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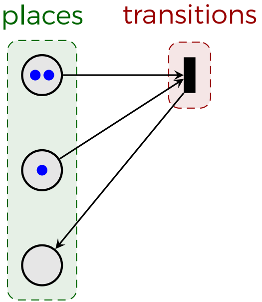


Petri nets are bipartite graphs.

https://en.wikipedia.org/wiki/Bipartite_graph

Static Properties

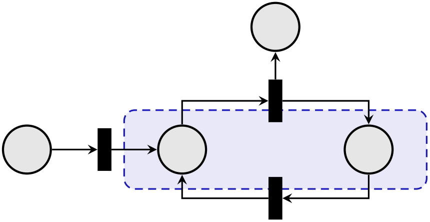
Properties depending only on the graph structure.

- independent of the dynamic states
- induced by loops, cycles, SCC, etc.

Properties that hold independently of the initial marking.

Traps

Trap = a subset of places S such that all transitions consuming tokens from S also put tokens into S.

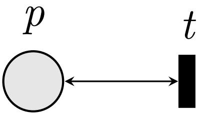


It's a trap!

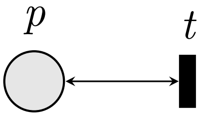
Once a trap contains tokens, it will always contain tokens.

Traps do not include transitions.

Where Is the Trap? $^1/_2$



Where Is the Trap? $1/2$



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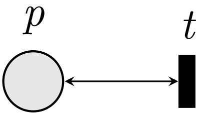


Answer: $\{p\}$

Where Is the Trap? $^2/_2$

https://en.wikipedia.org/wik

Where Is the Trap? $^2/_2$

https://en.wikipedia.org/wik

Where Is the Trap? $^2/_2$

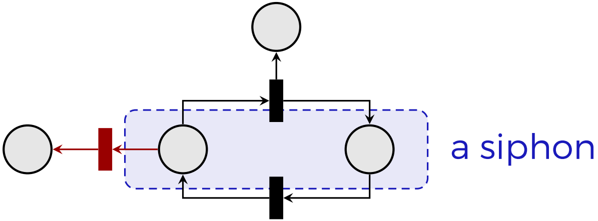
Answer: $\{p_1\}$, $\{p_2\}$, $\{p_1, p_2\}$

The property of being a trap is satisfied vacuously.

https://en.wikipedia.org/wik

Siphons

Siphon = a subset of places S such that all transitions putting tokens into S also consume tokens from S.



Siphons are duals (the opposite) of traps.

Once a siphon contains no tokens, it will never contain tokens again.

Where Is the Siphon?

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Where Is the Siphon?

Answer: $\{p_1\}$, $\{p_2\}$, $\{p_1, p_2\}$

Where Is the Siphon?

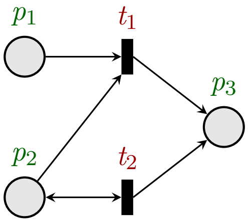
Answer: $\{p_1\}$, $\{p_2\}$, $\{p_1, p_2\}$

Can you think of other examples of siphons?

The incidence matrix M of a Petri net contains

- one row per place
- one column per transition

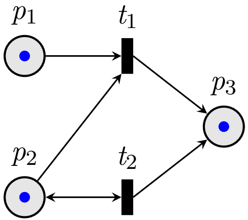
Cell (p, t) contains the value by which the number of tokens in p changes when t fires.



$M =$

	t_1	t_2
p_1	-1	0
p_2	-1	0
p_3	1	1

Petri Nets as Linear Operators: Dynamics

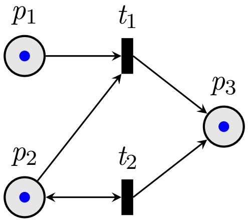


$$M = \begin{array}{c|cc} & t_1 & t_2 \\ \hline p_1 & -1 & 0 \\ p_2 & -1 & 0 \\ p_3 & 1 & 1 \end{array}$$

Current marking: $p_1^1\, p_2^1\, p_3^1 \quad \mapsto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

The firing vector: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$—$t_1$ fires once, t_2 does not fire

Petri Nets as Linear Operators: Dynamics



$M =$

	t_1	t_2
p_1	-1	0
p_2	-1	0
p_3	1	1

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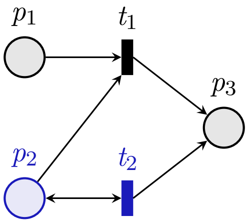
The firing vector: $\begin{pmatrix}1\\0\end{pmatrix}$—$t_1$ fires once, t_2 does not fire

Next marking:

$$\begin{pmatrix}1\\1\\1\end{pmatrix} + \begin{pmatrix}-1 & 0\\-1 & 0\\1 & 1\end{pmatrix} \cdot \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \begin{pmatrix}-1\\-1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\2\end{pmatrix}$$

Petri Nets $\neq$ Linear Operators



$$M = \begin{array}{c|cc} & t_1 & t_2 \\ \hline p_1 & -1 & 0 \\ p_2 & -1 & 0 \\ p_3 & 1 & 1 \end{array}$$

Note: t_2 cannot fire if p_2 is empty.

Cells (p_1, t_2), (p_2, t_2) both contain 0, but t_2 actually depends on the number of tokens in p_2!

- Petri nets cannot be completely reduced to linear operators.

Two Matrix-based Structural Properties

Transition invariant = a firing vector F such that $M \cdot F = 0$.

- F describes how to fire transitions such that the contents of the places does not change.
- for any marking!

Place invariant = a vector Y such that $M^T \cdot Y = 0$

- Existence of place invariants with all components non-negative $\implies$ conservation of tokens (like in chemistry).

Structural Properties and Behaviour

Structural properties = strong properties

- derived from the structure of the network
- holding for any possible state

For some types of Petri nets, behavioural properties can be described purely structurally:

- place invariants may describe boundedness
- traps and siphons may describe liveness

Structural properties are easier to handle.

- no need to look at the state graph

Summary of Structural Properties

- Traps
  - Once non-empty, always non-empty.
- Siphons
  - Once empty, always empty.
- Matrix-based
  - Properties of the incidence matrix.